Int. J. Solids Structures Vol. 17, pp. 741–751, 1981
Printed in Great Britain

0020-7683/81/080741-11$02.00/0
Pergamon Press Ltd.

AN APPROXIMATE THEORY OF SECONDARY CREEP FOR A CLASS OF THIN STRUCTURES†

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(*Received* 2 *October* 1980)

Abstract—The geometric assumption of thinness is applied to an integrodifferential equation which governs a class of secondary creep problems in such a way as to reduce it to a first order ordinary differential equation. Results are then obtained for solutions of the latter equation concerning monotonicity and asymptotic behavior and comparing them with solutions of the original equation at infinity. Also, for special cases, various closed form solutions are presented.

1. INTRODUCTION

It is well known that, in compound structures, creep effects are most important in those members which are in some sense thin, such as beams and plates. In the case of steady-state creep, the derivation of equations of thin objects such as plates and shells can, for many problems, be accomplished by using the techniques of linear elasticity in a somewhat modified form (see for example [1]). A difficulty arises in the case of transient creep if one wishes to include the effect of stress redistribution under constant loads. A powerful technique for dealing with this problem is the construction of an appropriate variational principle (see Rabotnov [2] Chapt. XI).

This paper considers a simple but technically important class of problems in which the introduction of the thinness approximation can be accomplished directly without need to resort to variational techniques and in such a way as to guarantee that the stress redistribution effect is included. Namely, these are the problems which are governed by eqn (3.1) of [3] (eqn (2.1) below). Included are cylindrical and spherical pressure vessels subject to internal pressure, beams with symmetric cross-sections undergoing pure bending and circular beams and thin strips under torsion. The assumed strain–stress law takes the form:

$$\epsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}] + \int_0^t F(\sigma_e)s_{ij}\,d\tau$$
$$s_{ij} = \sigma_{ij} - \frac{\delta_{ij}}{3}\sigma_{kk}, \quad \sigma_e = \sqrt{\left(\frac{3}{2}s_{ij}s_{ij}\right)}. \tag{1.1}$$

Roughly speaking, the technique involves approximating a quantity which is known to be constant in r at $t = \infty$ by a function which is linear in r for all $t > 0$. This can be done in such a way that (2.1), which is actually an integrodifferential equation in r and t, is replaced by a single first order ordinary differential equation (2.33). In the present era of programmable calculators, this is practically equivalent to solving the problem in closed form. Such an approximation would seem to be especially appropriate for problems such as the pure bending of beams and the torsion of thin strips in which a thin body assumption was used in the derivation of the original equation.

In Section 2, the basic ordinary differential equation (2.33) is derived. In order to achieve a single equation, it turns out that the two cases $a > 0$ and $a = 0$, where a is the lower limit of the space variable r, require separate treatment. The dependent variable $s(t)$ in (2.33) is actually $s(b, t)$. However, from it, other values $s(r, t)$ are readily obtained.

In Section 3, it is rigorously proved that $s(t)$ tends monotonically to a limit $s(\infty)$. In Section 4, this limit is compared to $s(b, \infty)$. It is shown that the ratio $s(\infty)/s(b, \infty)$ tends to unity as $h \to 0$ in the case $a > 0$. For $s = 0$, it is shown that, in most of the important special cases, the ratio is

†This research was supported by the National Science Foundation under Grant MCS 79-03393.

also independent of h. Since $s(0)/s(b,0)$ equals unity in all cases, it is plausible that $s(t)$ would furnish a good approximation for all time.

Section 5 contains several closed form solutions of (2.33), both for the case of a constant applied load and for unloading. Such solutions are common for steady-state creep, but are rare in the case of transient creep. The only example of which this author is aware occurs in [4], in which a collocation approximation is used in a beam problem.

The method used here for discretizing the spatial dependence of a function of space and time in order to obtain differential equations in time only is closely related to that employed by Einarsson [5] for the numerical integration of a certain class of integrodifferential equations which also have their origins in creep theory. Moreover, Einarsson also chooses his approximation so as to preserve asymptotic convergence. However, the present paper differs essentially from [5] both in point of view and in the form of the equations treated.

2. DERIVATION OF THE EQUATION

Consider the equation

$$s(r,t)=\frac{r^l}{I}\left(N(t)+\int_0^t\int_a^b H(s)q(\xi)\,d\xi\,d\tau\right)-\int_0^t H(s)\,d\tau \tag{2.1}$$

$$(a\le r\le b,\quad t\ge 0)$$

where

$$I=\int_a^b \xi^l q(\xi)\,d\xi. \tag{2.2}$$

It is assumed that

$$q(r)>0\ (0<r<a),\quad l\neq 0,\quad 0<I<\infty. \tag{2.3}$$

Also,

$$H(0)=0,\quad H'(z)>0,\quad H''(z)>0\ (0<z<\infty) \tag{2.4}$$

and H is odd on $(-\infty,\infty)$.

One special case of (2.1) is given by eqn (2.36) of [6],

$$\sigma(r,t)=\frac{\beta P(t)}{r^j}+u\left(\frac{\beta}{r^j}\int_a^b\int_0^t \sigma^n(\xi,\tau)\,d\tau\frac{d\xi}{\xi}-\int_0^t\sigma^n(r,\tau)\,d\tau\right) \tag{2.5}$$

which governs the problem of a cylindrical or spherical pressure vessel of inner radius a and outer radius b subject to a nondecreasing internal pressure. Here, the power law (1.1) has been assumed. If the restriction that the internal pressure be nondecreasing is removed, then (2.5) takes the form

$$\sigma(r,t)=\beta r^{-j}\left(P(t)+\mu\int_a^b\int_0^t|\sigma|^{n-1}\sigma\,d\tau\frac{d\xi}{\xi}\right)-\mu\int_0^t|\sigma|^{n-1}\sigma\,d\tau. \tag{2.6}$$

Here

$$\beta^{-1}=\int_a^b\xi^{-j-1}\,d\xi,\quad j=2(\text{cyl}),\quad j=3(\text{sphere}).$$

P is proportional to the internal pressure and σ to $\sigma_\theta-\sigma_r$. Thus (2.1) includes (2.6) if we take

$$s=\sigma,\quad l=-j,\quad I=\beta^{-1},\quad N=P,\quad H(z)=\mu|z|^{n-1}z,\quad q=\xi^{-1}. \tag{2.7}$$

For St-Venant pure bending problems, the equation analogous to (2.1) is (2.24) of [7] with $m=0$. In this case s is the tensile stress in that part of the beam ($a\le x\le b$) which is under tension, N equals $-M$, where M is the moment applied at the ends of the beam and $l=1$. Also, $a=0$ and corresponds to the center of the beam.

For a solid or hollow circular beam undergoing torsion, the governing equation is (2.18) of [3], again setting $m=0$. Here s is the shear stress $\sigma_{\theta z}$, N is proportional to the torsional moment and again $l=1$. The inner radius a can be either zero or positive depending on whether the beam is solid or hollow. Finally, we mention eqn (2.22) of [8] for the torsion of thin strips where, again, $l=1$ and $a=0$.

In order to motivate the approximation procedure, we take the formal limit of (2.1) as $t\to\infty$ under the assumptions that $N(t)\to N(\infty)$, $s(r,t)\to s(r,\infty)$, $\dot N(t)\to 0$, $\dot s(r,t)\to 0$ as $t\to\infty$. Here, a superposed dot denotes differentiation with respect to time. Thus, if (2.1) is differentiated with respect to time, and the limit is taken as $t\to\infty$, one obtains

$$H[s(r,\infty)]=\frac{r^l}{I}\int_a^b H[s(\xi,\infty)]q(\xi)\,\mathrm{d}\xi. \tag{2.9}$$

Since

$$r^{-l}H[s(r,\infty)]=C \quad \text{(constant)} \tag{2.10}$$

it is reasonable to assume that for $b-a\ll 1$, the quantity

$$\phi(r,t)\equiv r^{-l}\int_0^t H[s(r,\tau)]\,\mathrm{d}\tau \tag{2.11}$$

is *linear* in r.† Notice that, if one assumes that $\phi\equiv C(t)$ in [a, b] and plugs this into (2.1), the result is

$$s=\frac{r^lN}{I}$$

which is the elastic solution.

From this point on, it is convenient to set $b=a+h$. We write

$$\phi(\xi,t)=[\phi_2(t)-\phi_1(t)]\left(\frac{\xi-a}{h}\right)+\phi_1(t) \tag{2.12}$$

where

$$\phi_1(t)=\phi(a,t),\quad \phi_2(t)=\phi(a+h,t). \tag{2.13}$$

Therefore, by (2.11) and (2.12), the double integral in (2.1) has the approximate value

$$\begin{aligned} I^{-1}\int_0^t\int_a^b H(s)q(\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau &= I^{-1}\int_a^{a+h}\phi(\xi,t)\xi^lq(\xi)\,\mathrm{d}\xi,\\ &= c_1(h)\phi_1(t)+c_2(h)\phi_2(t), \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} c_1(h)&=h^{-1}[(a+h)-I^{-1}\int_a^{a+h}\xi^{l+1}q(\xi)\,\mathrm{d}\xi]\\ &=1-c_2(h)\equiv c(h). \end{aligned} \tag{2.15}$$

If (2.14) is inserted into (2.1) and the resulting equation is evaluated first at $r=a$, then at $r=a+h$, there results the following system of two equations in two unknowns:

$$\begin{aligned} s_1(t)=\frac{a^lN(t)}{I}&+a^l[c_1(h)\phi_1(t)+c_2(h)\phi_2(t)]-\\ &-\int_0^t H[s_1(\tau)]\,\mathrm{d}\tau, \end{aligned} \tag{2.16}$$

$$\begin{aligned} s_2(t)=\frac{(a+h)^lN(t)}{I}&+(a+h)^l[c_1(h)\phi_1(t)+c_2(h)\phi_2(t)]\\ &-\int_0^t H[s_2(\tau)]\,\mathrm{d}\tau. \end{aligned} \tag{2.17}$$

Here

$$s_1(t)=s(a,t),\quad s_2(t)=s(a+h,t). \tag{2.18}$$

†This leads to the same result as the assumption that $r^{-l}H[s(r,t)]$ is linear in r and is more concise.

Consider first the case $a=0$. In all such situations mentioned above, $l=1$. However, for greater generality it will only be required that $l>0$. It then follows from (2.4) and (2.11) that (2.16) is trivially satisfied by taking $s_1=0$ (this is also required physically) and that (2.17) becomes

$$s_2(t)=h^l I^{-1}N(t)+h^l c\phi_1-c\int_0^t H[s_2]\,d\tau. \tag{2.19}$$

However, at this point, a difficulty arises. For, assuming again that (2.8) holds with $N(\infty)>0$, it can be shown that the constant C in (2.10) is nonzero. Therefore, by (2.11) one must expect that $\phi_1\neq 0$ for sufficiently large t.

Consequently, we postpone this case and assume instead $a>0$, or, without loss of generality, $a=1$. One can henceforth think of the r variable as being nondimensional.

Notice that, by (2.15) and the first of (2.3),

$$\begin{aligned} c(h)&=h^{-1}\Big[(1+h)-I^{-1}\int_1^{1+h}\xi^{l+1}q(\xi)\,d\xi\Big]\\ &>h^{-1}[(1+h)-I^{-1}(1+h)I]=0,\\ c(h)&<h^{-1}[(1+h)-I^{-1}I]=1.\end{aligned}$$

Thus

$$0<c(h)<1. \tag{2.20}$$

Applying (2.11) and (2.15), one can rewrite the system (2.16), (2.17) in the form

$$s_1(t)=\frac{N}{I}+(1-c)(\phi_2-\phi_1) \tag{2.21}$$

$$s_2=(1+h)^l\Big[\frac{N}{I}+c(\phi_1-\phi_2)\Big]. \tag{2.22}$$

Eliminating the term $\phi_1-\phi_2$, we obtain

$$c(1+h)^l s_1+(1-c)s_2=(1+h)^l\frac{N}{I}. \tag{2.23}$$

The last two equations imply that

$$\begin{aligned} s_2=(1+h)^l\Big[\frac{N}{I}+c\int_0^t\Big(H\Big[\frac{N}{cI}-\frac{(1-c)s_2}{c(1+h)^l}\Big]\\ -(1+h)^{-l}H[s_2]\Big)\,d\tau\Big].\end{aligned} \tag{2.24}$$

For the case $a=0$, we approximate the r dependence of ϕ by a straight line through the points $(\lambda h,\phi_1)$, (h,ϕ_2), for some $0<\lambda<1$. The equation analogous to (2.12) is then

$$\phi(\xi,t)=[\phi_2(t)-\phi_1(t)]\frac{(\xi-\lambda h)}{h(1-\lambda)}+\phi_1(t).$$

It follows that

$$I^{-1}\int_0^t\int_0^h H(s)q(\xi)\,d\xi\,d\tau=c_1\phi_1+c_2\phi_2 \tag{2.25}$$

where

$$c_1(\lambda,h)=\frac{1-Q}{1-\lambda}=1-c_2\equiv c \tag{2.26}$$

and

$$Q=h^{-1}I^{-1}\int_0^h\xi^{l+1}q(\xi)\,d\xi. \tag{2.27}$$

Clearly $0<Q<1$, so that $c>0$. In order for $c<1$, one must make the restriction $0<\lambda<Q$.

If the approximation (2.25) is now inserted into (2.1) and the resulting equation is evaluated at $r=\lambda h$ and $r=h$, one obtains

$$s_1=(\lambda h)^l\left[\frac{N}{I}+(1-c)(\phi_2-\phi_1)\right], \tag{2.28}$$

$$s_2=h^l\left[\frac{N}{I}+c(\phi_1-\phi_2)\right] \tag{2.29}$$

where

$$s_1(t)=s(\lambda h,t),\quad s_2(t)=s(h,t), \tag{2.30}$$

and the identity

$$s_1+\frac{\lambda^l}{c}(1-c)s_2=(\lambda h)^l\frac{N}{Ic} \tag{2.31}$$

holds. It follows from (2.11), (2.29) and (2.31) that s_2 satisfies

$$s_2=\frac{h^lN}{I}+c\int_0^t\left(\lambda^{-1}H\left[(\lambda h)^l\frac{N}{cI}-\frac{\lambda^l}{c}(1-c)s_2\right]-H[s_2]\right)\mathrm{d}\tau. \tag{2.32}$$

Clearly, both (2.24) and (2.32) are special cases of the equation

$$s(t)=\alpha^{-1}\left[M(t)+c\int_0^t\left(H\left[\frac{M}{c}-(1-c)\frac{\alpha s}{c}\right]-\alpha H[s]\right)\mathrm{d}\tau\right].$$
$$(0<a,\quad 0<c<1) \tag{2.33}$$

With

$$\alpha=(1+h)^{-l},\quad M=\frac{N}{I},\quad s=s_2, \tag{2.34}$$

(2.33) becomes (2.24); with

$$\alpha=\lambda^l,\quad M=(\lambda h)^lNI^{-1} \tag{2.35}$$

it reduces to (2.32). It should be clear from the context as to when s refers to the function of r and t which satisfies (2.1) and when it refers to the solution of (2.33).

3. STRESS REDISTRIBUTION UNDER CONSTANT LOAD

For the stress redistribution problem, it is assumed that M is a positive constant. In this case,

$$s(0)=\alpha^{-1}M, \tag{3.1}$$

which implies that $s_2(0)=s(b,0)$ for either $a=1$ or $a=0$. To formally obtain $s(\infty)$, one applies to (2.33) the same reasoning that was used on (2.1) to get (2.9). It turns out that $s(\infty)$ must satisfy

$$H\left[\frac{M}{c}-(1-c)\frac{\alpha z}{c}\right]=\alpha H(z). \tag{3.2}$$

Notice that, since H satisfies (2.4), $M>0$ and $0<c<1$, any solution z of (3.2) must satisfy

$$0<z<\frac{M}{\alpha(1-c)}. \tag{3.3}$$

Consider the function

$$F(z)=H\left[\frac{M}{c}-(1-c)\frac{\alpha z}{c}\right]-\alpha H(z).$$

Since

$$F'(z)=-(1-c)\frac{\alpha}{c}H'\left[\frac{M}{c}-(1-c)\frac{\alpha z}{c}\right]-\alpha H'(z)<0 \tag{3.4}$$

for $-\infty < z < \infty$, the solution of (3.2) (easily shown to exist) is unique. In the case where H is positively homogeneous of degree n, i.e. $H(\lambda z) = \lambda^n H(z)$, $(\lambda > 0, -\infty < z < \infty)$, it takes the form

$$z = M[(1-c)\alpha + c\alpha^{1/n}]^{-1}. \tag{3.5}$$

Assuming that a differentiable solution $s(t)$ of (2.33) exists on $(0, \infty)$, let us now rigorously examine its behavior as $t \to \infty$. We first notice that, for M constant, (2.33) is equivalent to the initial value problem

$$\dot{s} = \alpha^{-1} c F(s) \quad (t > 0), \tag{3.6}$$

$$s(0) = \alpha^{-1} M. \tag{3.7}$$

Thus

$$\dot{s}(0) = \alpha^{-1} c F[s(0)] = \alpha^{-1} c (H[M] - \alpha H[\alpha^{-1} M]). \tag{3.8}$$

Since by (2.4) H is strictly convex, $\dot{s}(0) \neq 0$, so that $s(t)$ is nonconstant. In fact, as shown in [3] eqn (3.16), (2.4) implies that

$$H(\mu x) > \mu H(x) \quad (\mu > 1, \quad x > 0). \tag{3.9}$$

Applying this result to (3.8), we see that

$$\dot{s}(0) < 0 \quad \text{for} \quad \alpha < 1, \quad \dot{s}(0) > 0 \quad \text{for} \quad \alpha > 1. \tag{3.10}$$

Notice that $\alpha > 1$ corresponds to the case of the pressure vessel, while for the torsion and pure bending problems $\alpha < 1$. Therefore, (3.10) agrees with known stress redistribution behavior for these cases.

Suppose that, at some time $t_1 > 0$, $\dot{s}(t_1) = 0$. Then by (3.6), $s(t_1) = z$, where z is the solution of (3.2). However, the constant function $u(t) = z$ satisfies the initial value problem

$$\dot{u} = \alpha^{-1} c F(u) \quad (t > 0),$$

$$u(t_1) = z.$$

Consequently, by a standard uniqueness theorem in the theory of ordinary differential equations, $s(t) \equiv z$. Since, by (3.10), s cannot be constant, this means that s is a strictly monotone function of t, increasing for $\alpha > 1$ and decreasing for $\alpha < 1$. Also, at no finite time t can $s(t)$ be a zero of F.

Let $\alpha > 1$, and let z be the root of (3.2). Applying (3.9) to the right hand side of (3.2), one finds that

$$H\left[\frac{M}{c} - (1-c)\frac{\alpha z}{c}\right] < H[\alpha z].$$

Therefore, due to the monotonicity of H,

$$z > \alpha^{-1} M = s(0).$$

Consequently, z must be an upper bound for the set $\{s(t): t \geq 0\}$. This implies that the limit $s(\infty)$ exists and $s(\infty) \leq z$.

In fact, one can prove that $s(\infty) = z$.† For, by (3.4), (3.6),

$$\ddot{s} = \alpha^{-1} c F'(s) \dot{s} < 0 \quad (t > 0),$$

so that $\dot{s}(\infty) \equiv \lim_{t \to \infty} \dot{s}(t)$ must exist. Since $s(t)$ is bounded, $\dot{s}(\infty)$ must be zero. Therefore, taking the limit of (3.6) as $t \to \infty$, we find that $F[s(\infty)] = 0$.

†We are indebted to Prof. Platon Deliyannis for this observation.

Summarizing, we have found that, for $\alpha > 1$, $s(t)$ is monotone increasing and

$$\alpha^{-1}M = s(0) < s(t) < s(\infty) \quad (t > 0) \tag{3.11}$$

where $s(\infty)$ is the solution of (3.2). By analogous arguments, it follows that, if $\alpha < 1$, then $s(t)$ is monotone decreasing, and

$$s(\infty) < s(t) < \alpha^{-1}M \quad (t > 0). \tag{3.12}$$

4. COMPARISON OF EXACT AND APPROXIMATE SOLUTIONS

Since $s(0) = s(b, 0)$, if we can show that $s(\infty)$ closely approximates $s(b, \infty)$ we will have some encouragement in the belief that $s(t)$ uniformly approximates $s(b, t)$ on $(0, \infty)$. Clearly, having evaluated the constant C in eqn (2.10), one can obtain an expression for $s(r, \infty)$. To this end, let us multiply both sides of (2.1) by $q(r)$ and integrate the resulting equation from a to b. It follows that

$$\int_a^b s(r, t)q(r)\,\mathrm{d}r = N.$$

Letting $t \to \infty$ and applying (2.10) we get

$$\int_a^b H^{-1}(r^l C)q(r)\,\mathrm{d}r = N.$$

For the remainder of this section, it is assumed that H is positively homogeneous of degree $n > 1$. Then H^{-1} is positively homogeneous of degree $1/n$, so that

$$H^{-1}(C) = N \Big/ \int_a^b \xi^{l/n} q(\xi)\,\mathrm{d}\xi$$

and†

$$s(r, \infty) = \frac{N r^{l/n}}{\int_a^b \xi^{l/n} q(\xi)\,\mathrm{d}\xi}. \tag{4.1}$$

$s(\infty)$ is, of course, given by (3.5). We wish to compute

$$\lim_{h \to 0} \frac{s(\infty)}{s(b, \infty)} = \lim_{h \to 0} \frac{M \int_a^b \xi^{l/n} q(\xi)\,\mathrm{d}\xi}{N b^{l/n}[(1-c)\alpha + c\alpha^{1/n}]}.$$

Let us first consider the case $a = 1$, $b = 1 + h$. Then (2.34) applies, and

$$\frac{s(\infty)}{s(b, \infty)} = I^{-1} \int_1^{1+h} \xi^{l/n} q(\xi)\,\mathrm{d}\xi[(1-c)(1+h)^{l((1/n)-1)} + c]^{-1}.$$

By (2.2) and (2.15),

$$\lim_{h \to 0} I^{-1} \int_1^{1+h} \xi^{l/n} q(\xi)\,\mathrm{d}\xi = 1$$

and

$$\lim_{h \to 0} c = \lim_{h \to 0} h^{-1}\left[(1+h) - I^{-1} \int_1^{1+h} \xi^{l+1} q(\xi)\,\mathrm{d}\xi\right] = \frac{1}{2}.$$

Therefore,

$$\lim_{h \to 0} \frac{s(\infty)}{s(1+h, \infty)} = 1. \tag{4.2}$$

†Actually, the second limit requires the additional assumption $q(1) \neq 0$, which is satisfied for all cases of physical interest.

In the case $a = 0$, $b = h$, it follows from (4.1), (3.5) and (2.35) that

$$\frac{s(\infty)}{s(h,\infty)} = I^{-1}h^{l(1-(1/n))}\int_0^h \xi^{l/n}q(\xi)\,d\xi\ [1-c+c\lambda^{l((1/n)-1)}]^{-1}. \tag{4.3}$$

To simplify the analysis, we add the restrictions

$$q(\xi) = q_0\xi^{\nu}, \quad l = 1, \quad q_0 > 0, \quad \nu > 0. \tag{4.4}$$

Included in this class of problems is the torsion of thin strips and solid circular rods and the pure bending of beams with a rectangular cross-section. Then, using (2.26) and (2.27) we can put (4.3) in the form

$$\frac{s(\infty)}{s(h,\infty)} = \frac{Q_0(1-\lambda)\lambda^{(n-1)/n}}{(Q-\lambda)\lambda^{(n-1)/n}+1-Q}, \tag{4.5}$$

where

$$Q = h^{-1}I^{-1}\int_0^h \xi^{l+1}q(\xi)\,d\xi = \frac{2+\nu}{3+\nu} \tag{4.6}$$

$$Q_0 = I^{-1}h^{l(1-(1/n))}\int_0^h \xi^{l/n}q(\xi)\,d\xi = \frac{\nu+2}{\nu+1/n+1}. \tag{4.7}$$

From (4.3), it is immediate that

$$\frac{s(\infty)}{s(h,\infty)} = 1 \tag{4.8}$$

for $n = 1$ and any $0 < \lambda < Q$. We wish to show that there exists $0 < \lambda < Q$ such that (4.8) also holds for $n > 1$, i.e. that the equation

$$f(\lambda) \equiv \frac{Q_0(1-\lambda)\lambda^{(n-1)/n}}{(Q-\lambda)\lambda^{(n-1)/n}+1-Q} = 1$$

has a solution in $(0, Q)$. Since $f(0) = 0$, it suffices to show that

$$f(Q) = Q_0Q^{(n-1)/n} > 1. \tag{4.9}$$

But by (4.6), (4.7)

$$Q_0Q^{(n-1)/n} = \frac{\nu+2}{\nu+1/n+1}\left(\frac{2+\nu}{3+\nu}\right)^{(n-1)/n}$$
$$= \frac{(\nu+2)^2}{(\nu+3)(\nu+1/n+1)}\left(\frac{\nu+3}{\nu+2}\right)^{1/n}.$$

Let

$$g(x) = \frac{(\nu+2)^2}{(\nu+3)(\nu+x+1)}\left(\frac{\nu+3}{\nu+2}\right)^{x}.$$

Since $g(0) > 1$ and $g(1) = 1$, (4.9) holds for all $n > 1$ provided $g'(x) < 0$ for $0 < x < 1$. But,

$$g'(x) = \left(\frac{\nu+3}{\nu+2}\right)^{x}\frac{(\nu+2)^2}{(\nu+3)(\nu+x+1)}\left[\ln\left(\frac{\nu+3}{\nu+2}\right) - \frac{1}{\nu+x+1}\right]$$

and

$$\ln\left(\frac{\nu+3}{\nu+2}\right) < \frac{2\nu+5}{2(\nu+2)(\nu+3)} < \frac{1}{\nu+2} < \frac{1}{\nu+x+1}.$$

Here we have used the secant line upper bound

$$\ln z < \frac{z^2-1}{2z} \quad (z > 1).$$

5. CLOSED FORM SOLUTIONS FOR CONSTANT M

In this section, we consider the simplest non-linear creep law,

$$H(z) = K\,|\,z\,|\,z. \tag{5.1}$$

For M constant, (2.33) is equivalent to the initial value problem

$$\dot{s} = \alpha^{-1}c\left(H\left[\frac{M}{c} - (1-c)\frac{\alpha s}{c}\right] - \alpha H[s]\right), \tag{5.2}$$

$$s(0) = \frac{M}{\alpha}. \tag{5.3}$$

By (5.1) and (3.5), the inequalities (3.11) and (3.12) become, respectively,

$$\alpha^{-1}M < s(t) < M([1-c]\alpha + c\alpha^{1/2})^{-1} \quad (\alpha > 1), \tag{5.4}$$

$$\alpha^{-1}M > s(t) > M([1-c]\alpha + c\alpha^{1/2})^{-1} \quad (\alpha < 1). \tag{5.5}$$

In either case,

$$\frac{M}{c} - (1-c)\frac{\alpha s}{c} > M \quad (t > 0).$$

Therefore, (5.2) takes the form

$$\dot{s} = \frac{Kc}{\alpha}\left(\left[\frac{M}{c} - (1-c)\frac{\alpha s}{c}\right]^2 - \alpha s^2\right),$$

$$= \frac{Kc}{\alpha}\left(\frac{M^2}{c^2} - \frac{2M}{c^2}(1-c)\alpha s + \left[\left(\frac{1-c}{c}\right)^2\alpha - 1\right]\alpha s^2\right). \tag{5.6}$$

An important special case occurs when

$$\alpha = \left(\frac{c}{1-c}\right)^2. \tag{5.7}$$

This condition is actually realized for hollow circular cylinders subject to internal pressure. The solution of (5.2), (5.3) is then

$$s(t) = \frac{M}{2(1-c)\alpha}\left\{1 + (1-2c)\exp\left[-\frac{2}{c}(1-c)KMt\right]\right\}. \tag{5.8}$$

Notice that, by virtue of (5.7), s is increasing for $\alpha > 1$ and decreasing for $\alpha < 1$, as was predicted in section 3. It is of interest that, for increasing M not only is the magnitude of s increased, but also the rate of stress redistribution.

In the general case $\alpha \neq c^2/(1-c)^2$, (5.6) becomes

$$\dot{s} = -A_1\left[1 - \left(\frac{s - A_2}{A_3}\right)^2\right] \tag{5.9}$$

which also has a closed form solution. Here,

$$A_1 = \frac{KcM^2}{\alpha[(1-c)^2\alpha - c^2]},$$

$$A_2 = \frac{M(1-c)}{(1-c)^2\alpha - c^2}, \tag{5.10}$$

$$A_3 = \frac{cM}{\alpha^{1/2}[(1-c)^2\alpha - c^2]}.$$

Let

$$u = \frac{s - A_2}{A_3}.$$

Then (5.9) takes the form

$$\frac{\dot{u}}{1 - u^2} = \frac{-A_1}{A_3}. \tag{5.11}$$

By (5.10), (3.5) and (5.3),

$$u(0) = -[(1-c)\alpha^{1/2} + c\alpha^{-1/2}], \quad u(\infty) = -1.$$

For the sake of definiteness, suppose $c > 1/2$. Then $u(0) < -1$ for any $0 < \alpha < 1$. Therefore, since u is monotone, $|u(t)| > 1$ for all $t > 0$ and the solution of the initial value problem (5.2), (5.3) takes the form

$$s(t) = A_2 - A_3 \coth\left(\frac{A_1}{A_3} t + \operatorname{argcoth}[(1-c)\alpha^{1/2} + c\alpha^{-1/2}]\right). \tag{5.12}$$

Another interesting class of closed form solutions arises in the case of unloading. Suppose $M(t) = 0$ for $t > t_0$. Then (5.2) becomes

$$\dot{s} = -\alpha^{-1} c\left(H\left[(1-c)\frac{\alpha s}{c}\right] + \alpha H[s]\right) \quad (t > t_0) \tag{5.13}$$

which, for any power law

$$H(z) = K|z|^{n-1} z, \tag{5.14}$$

always has a closed form solution. Suppose, for example, that $s(t_0+) > 0$. Then in some interval (t_0, t_1), (5.13) has the form

$$\dot{s} = -Kc\left[\left(\frac{1-c}{c}\right)^n \alpha^{n-1} + 1\right] s^n \equiv -qs^n.$$

Therefore, for t in (t_0, t_1),

$$s(t) = [(n-1)q(t - t_0) + s^{1-n}(t_0+)]^{-1/(n-1)}. \tag{5.15}$$

It follows that $t_1 = \infty$. An interesting feature of this result is the algebraic damping of the stress in the case of unloading as compared with the exponential redistribution for a constant nonzero load.

6. CONCLUSION

A "thin body" differential equation (2.33) has been derived which governs transient creep for a class of technically important problems. It has been shown mathematically that solutions of this equation approximate or equal exact solutions at times zero and infinity and have the appropriate stress redistribution behavior. Thus (2.33) can be applied directly to creep problems.

It can also be used for comparison with other thin body theories, such as [2] and those described in [9].

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